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Bäcklund transformations and Lax pairs for two differential–difference equations

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Abstract
Two bilinear differential–difference equations are proposed in this paper. The corresponding bilinear Bäcklund transformations (BTs) are presented. Starting from the bilinear BTs, soliton solutions of the proposed equations are generated. By dependent variable transformations, these two equations are transformed into equations in nonlinear variables and the corresponding Lax pairs are derived.

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1. Introduction

The purpose of this short paper is to propose two new integrable differential–difference equations. It is well-known that there are many approaches to search for new integrable systems in the literature. Two effective ways to do so are via Hirota's method and Bäcklund transformations (BTs) [1–6]. Specifically, one first starts from some types of Hirota's bilinear equations and then test them for existence of multi-soliton solutions or bilinear BTs [7, 8]. Following [8], we consider a generalized bilinear differential–difference equation

$$F(D_x, D_t, \sinh(\alpha_1 D_n), \ldots, \sinh(\alpha_l D_n))\, f(n) \bullet f(n) = 0 \qquad (1)$$

where F is an even order polynomial in D_x, D_t, $\sinh(\alpha_1 D_n)$, ... and $\sinh(\alpha_l D_n)$, and l is a given positive integer; the α_i, where $i = 1, 2, \ldots, l$, are l different constants, and $F(0, 0, \ldots, 0) = 0$. Here Hirota's bilinear differential operator $D_x^m D_t^k$ and the bilinear difference operator $\exp(\delta D_n)$ are defined by [1–4]

$$D_x^m D_t^k\, a \bullet b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(x,t)b(x',t')|_{x'=x,t'=t}$$

$$\exp(\delta D_n)a(n)\bullet b(n) \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right] a(n)b(n')|_{n'=n} = a(n+\delta)b(n-\delta).$$

It is known that several celebrated lattices are of type (1). For examples, the two-dimensional Toda lattice

$$\frac{\partial^2 Q_n}{\partial x \partial y} = \exp(Q_{n-1} - Q_n) - \exp(Q_n - Q_{n+1}) \tag{2}$$

can be written as

$$[D_x D_y - 4\sinh^2(\tfrac{1}{2}D_n)]f(n)\bullet f(n) = 0 \tag{3}$$

by the dependent variable transformation $Q_n = \ln(f(n)/f(n+1))$, while the so-called Lotka–Volterra (LV) or Kac–van Moerbeke equation

$$u_t(n) = u(n)(u(n-1) - u(n+1)) \tag{4}$$

is transformed into bilinear form

$$[D_t \sinh(\tfrac{1}{2}D_n) + \cosh(\tfrac{3}{2}D_n) - \cosh(\tfrac{1}{2}D_n)]f(n)\bullet f(n) = 0 \tag{5}$$

through the dependent variable transformation

$$u(n) = \frac{f(n-\frac{3}{2})f(n+\frac{3}{2})}{f(n-\frac{1}{2})f(n+\frac{1}{2})}.$$

Let us recall that a simple procedure [8] for finding new integrable candidates of type (1) is to search for suitable F, G, A and B such that the relation

$$\begin{aligned}
&[F(D_x, D_t, \sinh(\alpha_1 D_n), \ldots, \sinh(\alpha_l D_n))f(n)\bullet f(n)]\\
&\qquad\times[G(D_x, D_t, \sinh(\beta_1 D_n), \ldots, \sinh(\beta_{s_1} D_n))g(n)\bullet g(n)]\\
&\quad= [F(D_x, D_t, \sinh(\alpha_1 D_n), \ldots, \sinh(\alpha_l D_n))g(n)\bullet g(n)]\\
&\qquad\times[G(D_x, D_t, \sinh(\beta_1 D_n), \ldots, \sinh(\beta_{s_1} D_n))f(n)\bullet f(n)]
\end{aligned}$$

can be derived from

$$A(D_x, D_t, \exp(\gamma_1 D_n), \ldots, \exp(\gamma_{s_2} D_n))f(n)\bullet g(n) = 0 \tag{6}$$

$$B(D_x, D_t, \exp(\omega_1 D_n), \ldots, \exp(\omega_{s_3} D_n))f(n)\bullet g(n) = 0 \tag{7}$$

where s_i, $i = 1, 2, 3$, are given positive integers and β_i, $i = 1, \ldots s_1$, γ_j, $j = 1, \ldots, s_2$, and ω_k, $k = 1, \ldots, s_3$, are constants. In this circumstance, (6) and (7) may be viewed as a BT for (1) if $G(D_x, D_t, \sinh(\beta_1 D_n), \ldots, \sinh(\beta_{s_l} D_n))f(n)\bullet f(n) \neq 0$. In the following we will report two new bilinear differential–difference equations of type (1) found in this context. One is

$$[D_x D_t \cosh(\tfrac{1}{2}D_n) + D_x \sinh(\tfrac{1}{2}D_n) + D_t \sinh(\tfrac{1}{2}D_n)]f(n)\bullet f(n) = 0 \tag{8}$$

and the other is

$$[D_t \sinh(\tfrac{1}{2}D_n) + D_x^3 \sinh(\tfrac{1}{2}D_n)]f(n)\bullet f(n) = 0. \tag{9}$$

It turns out that equation (8) can be transformed into

$$\begin{aligned}
&(u(n+1) + u(n))_{xt} + u_x(n+1)u_t(n+1) - u_x(n)u_t(n) + (u(n+1) - u(n))_x\\
&\qquad+ (u(n+1) - u(n))_t = 0
\end{aligned} \tag{10}$$

by the dependent variable transformation $u(n) = \ln(f(n+1)/f(n))$, whereas (9) becomes

$$
\begin{aligned}
&(u(n+1)-u(n))_t + 3(u(n+1)-u(n))_x(u(n+1)-u(n))^2 + (u(n+1)-u(n))_{xxx}\\
&\qquad +3(u(n+1)+u(n))_x(u(n+1)-u(n))_x\\
&\qquad +3(u(n+1)+u(n))_{xx}(u(n+1)-u(n)) = 0
\end{aligned}
\tag{11}
$$

through the dependent variable transformation $u(n) = (\ln f(n))_x$. We will also show that (10) and (11) are integrable in the sense of having Lax pairs.

This paper is organized as follows. In section 2, we will present a bilinear BT for (8). As a result, a Lax pair for (10) is derived from the BT. A bilinear BT for (9) is obtained in section 3. A Lax pair for (11) is also given. These results are summarized in section 4. Finally, in the appendix we list some bilinear operator identities used in the paper.

2. Bilinear BT for (8) and Lax pair for (10)

In this section, we will show that (8) is integrable in the sense of having a bilinear BT. In fact, we have the following result.

Proposition 1. *The bilinear equation (8) has the BT*

$$[D_t e^{\frac{1}{2}D_n} - \lambda D_t e^{-\frac{1}{2}D_n}] f(n)\bullet g(n) = [(\mu-2)e^{\frac{1}{2}D_n} - \lambda\mu e^{-\frac{1}{2}D_n}] f(n)\bullet g(n) \tag{12}$$

$$[\lambda D_x e^{-\frac{1}{2}D_n} + D_x e^{\frac{1}{2}D_n}] f(n)\bullet g(n) = [\gamma e^{\frac{1}{2}D_n} + (2\lambda+\lambda\gamma)e^{-\frac{1}{2}D_n}] f(n)\bullet g(n) \tag{13}$$

where λ, μ and γ are arbitrary constants.

This result can be proved by using Hirota's bilinear operator identities. We omit the details of the proof. Instead we are going to construct soliton solutions of (8) by using the BT (12) and (13). Firstly, by applying the BT (12) and (13) to the trivial solution $f(n) = 1$, we can obtain the 1-soliton solution

$$g(n) = 1 + \exp\left(pn + qx + rt + \eta^0\right)$$

for the parameters $\mu = 2/(1-\lambda)$, $\gamma = -2\lambda/(1+\lambda)$, where p, λ and η^0 are constants and

$$q = -\frac{2\lambda(e^{\frac{1}{2}p} - e^{-\frac{1}{2}p})}{(1+\lambda)(\lambda e^{\frac{1}{2}p} + e^{-\frac{1}{2}p})} \qquad r = \frac{2\lambda(e^{-\frac{1}{2}p} - e^{\frac{1}{2}p})}{(1-\lambda)(\lambda e^{\frac{1}{2}p} - e^{-\frac{1}{2}p})}.$$

Furthermore, by applying the BT (12) and (13) to the 1-soliton solution $f(n) = 1 + \exp(\eta_1)$, we can deduce the following 2-soliton solution

$$g(n) = 1 + A_1 e^{\eta_1} + e^{\eta_2} + A_2 e^{\eta_1+\eta_2}$$

where

$$
\begin{aligned}
\eta_i &= p_i n + q_i x + r_i t + \eta_i^0\\
A_1 &= \frac{[2\lambda_2 - r_1(1-\lambda_2)]e^{\frac{1}{2}p_1} + [\lambda_2 r_1(1-\lambda_2) - 2\lambda_2]e^{-\frac{1}{2}p_1}}{[2\lambda_2 + \lambda_2 r_1(1-\lambda_2)]e^{\frac{1}{2}p_1} - [r_1(1-\lambda_2) + 2\lambda_2]e^{-\frac{1}{2}p_1}}\\
A_2 &= \frac{[(r_1-r_2)(1-\lambda_2) - 2\lambda_2]e^{\frac{1}{2}(p_1-p_2)} + [2\lambda_2 - \lambda_2(r_1-r_2)(1-\lambda_2)]e^{-\frac{1}{2}(p_1-p_2)}}{-[\lambda_2(r_1+r_2)(1-\lambda_2) + 2\lambda_2]e^{\frac{1}{2}(p_1+p_2)} + [2\lambda_2 + (r_1+r_2)(1-\lambda_2)]e^{-\frac{1}{2}(p_1+p_2)}}\\
q_i &= -\frac{2\lambda_i(e^{\frac{1}{2}p_i} - e^{-\frac{1}{2}p_i})}{(1+\lambda_i)(\lambda_i e^{\frac{1}{2}p_i} + e^{-\frac{1}{2}p_i})} \qquad r_i = \frac{2\lambda_i(e^{-\frac{1}{2}p_i} - e^{\frac{1}{2}p_i})}{(1-\lambda_i)(\lambda_i e^{\frac{1}{2}p_i} - e^{-\frac{1}{2}p_i})}
\end{aligned}
$$

with p_i, λ_i and η_i^0 being constants for the set of parameters $\lambda = \lambda_2, \mu = 2/(1-\lambda_2)$ and $\gamma = -2\lambda_2/(1+\lambda_2)$. Besides, by using MATHEMATICA, we can show that (8) has the 3-soliton solutions

$$f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{12}A_{23}A_{13}e^{\eta_1+\eta_2+\eta_3}$$

where

$$\eta_i = p_i n + q_i x + r_i t + \eta_i^0$$

$$q_i = -\frac{2\lambda_i(e^{\frac{1}{2}p_i} - e^{-\frac{1}{2}p_i})}{(1+\lambda_i)(\lambda_i e^{\frac{1}{2}p_i} + e^{-\frac{1}{2}p_i})} \qquad r_i = \frac{2\lambda_i(e^{-\frac{1}{2}p_i} - e^{\frac{1}{2}p_i})}{(1-\lambda_i)(\lambda_i e^{\frac{1}{2}p_i} - e^{-\frac{1}{2}p_i})}$$

$$A_{ij} = -\frac{(q_i - q_j)(r_i - r_j)\cosh(\frac{1}{2}(p_i - p_j)) + (q_i - q_j + r_i - r_j)\sinh(\frac{1}{2}(p_i - p_j))}{(q_i + q_j)(r_i + r_j)\cosh(\frac{1}{2}(p_i + p_j)) + (q_i + q_j + r_i + r_j)\sinh(\frac{1}{2}(p_i + p_j))}.$$

Next, we want to derive a nonlinear superposition formula for (8). Suppose that

$$f_0(n) \stackrel{(\lambda_1,\mu_1,\gamma_1)}{\longrightarrow} f_1(n) \stackrel{(\lambda_2,\mu_2,\gamma_2)}{\longrightarrow} f_{12}(n)$$

and

$$f_0(n) \stackrel{(\lambda_2,\mu_2,\gamma_2)}{\longrightarrow} f_2(n) \stackrel{(\lambda_1,\mu_1,\gamma_1)}{\longrightarrow} f_{21}(n).$$

Based on the permutability of the BT (12) and (13), i.e. $f_{12}(n) = f_{21}(n)$, and with some calculations, we can derive the following nonlinear superposition formula:

$$\left(e^{-\frac{1}{2}D_n} - \frac{1}{\lambda_1\lambda_2}e^{\frac{1}{2}D_n}\right) f_0 \bullet f_{12} = c\big(\lambda_1 e^{\frac{1}{2}D_n} - \lambda_2 e^{-\frac{1}{2}D_n}\big) f_1 \bullet f_2$$

with c being a nonzero constant, which is similar to that for a differential–difference CDGKS equation [9].

In the following, we are going to derive a Lax pair for (10) from the BT (12) and (13). For this purpose, we set $\psi(n) = g(n)/f(n), u(n) = \ln(f(n+1)/f(n))$. Then from the bilinear BT (12) and (13), we have

$$(\lambda\psi(n+1) - \psi(n))_t = -u_t(n)(\psi(n) + \lambda\psi(n+1)) + \mu(\psi(n) - \lambda\psi(n+1)) - 2\psi(n) \quad (14)$$

$$-(\lambda\psi(n+1) + \psi(n))_x = u_x(n)(\lambda\psi(n+1) - \psi(n)) + \gamma\psi(n) + (2\lambda + \lambda\gamma)\psi(n+1). \quad (15)$$

Furthermore we introduce $T_1 = \lambda T_+ - 1$, $T_2 = 1 + \lambda T_+$ and $\psi_1 = T_1\psi(n)$, $\psi_2 = T_2\psi(n)$ where T_+ is a shift operator defined by $T_+u(n) \equiv u(n+1)$. Then (14) and (15) become

$$\psi_{1t} = -(u_t(n) + 1)\psi_2 + (1-\mu)\psi_1 \quad (16)$$

$$\psi_{2x} = -(u_x(n) + 1)\psi_1 - (\gamma + 1)\psi_2. \quad (17)$$

By some calculations and using the relations

$$T_1\psi_2 = T_2\psi_1$$

$$T_1(a(n)b(n)) = \tfrac{1}{2}(b(n+1) - b(n))T_2a(n) + \tfrac{1}{2}(b(n+1) + b(n))T_1a(n)$$

$$T_2(a(n)b(n)) = \tfrac{1}{2}(b(n+1) + b(n))T_2a(n) + \tfrac{1}{2}(b(n+1) - b(n))T_1a(n)$$

we can derive (10) from the compatibility condition: $(T_1\psi_2)_{xt} = (T_2\psi_1)_{tx}$. The result can be summarized as follows:

Proposition 2. *Equations (14) and (15) constitute a Lax pair for (10).*

Furthermore, concerning bilinear equation (8), we have found the following Lie symmetries:

$$\sigma_1 = (h_1(t) + h_2(x))f$$

$$\sigma_2 = h_1(t)f_t + (n-x)h_1(t)f$$

$$\sigma_3 = h_2(x)f_x + (n-t)h_2(x)f$$

where $h_1(t)$ and $h_2(x)$ are arbitrary functions of t and x respectively.

3. Bilinear BT for (9) and Lax pair for (11)

We now turn to consider the bilinear equation (9). Concerning (9), we have the following result:

Proposition 3. *The bilinear equation (9) has the BT*

$$D_x e^{\frac{1}{2}D_n} f(n)\bullet g(n) = (\lambda D_x e^{-\frac{1}{2}D_n} + \mu e^{\frac{1}{2}D_n} - \lambda\mu e^{-\frac{1}{2}D_n}) f(n)\bullet g(n) \tag{18}$$

$$(D_t - 3\mu D_x^2 + D_x^3 + 3\mu^2 D_x + \gamma) f(n)\bullet g(n) = 0 \tag{19}$$

where λ, μ and γ are arbitrary constants.

Proof. Let $f(n)$ be a solution of equation (9). If we can show that equations (18) and (19) guarantee that the following relation

$$P \equiv [D_t \sinh(\tfrac{1}{2}D_n) + D_x^3 \sinh(\tfrac{1}{2}D_n)]g(n)\bullet g(n) = 0$$

hold, then equations (18) and (19) form a Bäcklund transformation. □

In fact, by equations (A.1)–(A.5), (18) and (19), we have

$$\begin{aligned}
-[e^{\frac{1}{2}D_n} f(n)\bullet f(n)]P &= 2\sinh(\tfrac{1}{2}D_n)[(D_t + D_x^3) f(n)\bullet g(n)]\bullet f(n)g(n) \\
&\quad -3D_x(D_x e^{\frac{1}{2}D_n} f(n)\bullet g(n))\bullet(D_x e^{-\frac{1}{2}D_n} f(n)\bullet g(n)) \\
&= 2\sinh(\tfrac{1}{2}D_n)[(D_t + D_x^3) f(n)\bullet g(n)]\bullet f(n)g(n) \\
&\quad -3D_x[(\mu e^{\frac{1}{2}D_n} - \lambda\mu e^{-\frac{1}{2}D_n}) f(n)\bullet g(n)]\bullet(D_x e^{-\frac{1}{2}D_n} f(n)\bullet g(n)) \\
&= 2\sinh(\tfrac{1}{2}D_n)[(D_t + D_x^3) f(n)\bullet g(n)]\bullet f(n)g(n) \\
&\quad -3\mu[2\sinh(\tfrac{1}{2}D_n)(D_x^2 f(n)\bullet g(n))\bullet f(n)g(n) \\
&\quad -D_x(D_x e^{\frac{1}{2}D_n} f(n)\bullet g(n))\bullet(D_x e^{\frac{1}{2}D_n} f(n)\bullet g(n))] \\
&\quad -\lambda\mu D_x(e^{-\frac{1}{2}D_n} f(n)\bullet g(n))\bullet(D_x e^{-\frac{1}{2}D_n} f(n)\bullet g(n)) \\
&= 2\sinh(\tfrac{1}{2}D_n)[(D_t - 3\mu D_x^2 + D_x^3) f(n)\bullet g(n)]\bullet f(n)g(n) \\
&\quad +6\mu^2 \sinh(\tfrac{1}{2}D_n)(D_x f(n)\bullet g(n))\bullet f(n)g(n) = 0.
\end{aligned}$$

In this way we have completed the proof of proposition 3.

We are going to construct soliton solutions of (9) by using the BT (18) and (19). Firstly, by applying the BT (18) and (19) to the trivial solution $f(n) = 1$, we can obtain the 1-soliton solution

$$g(n) = 1 + \exp\left(pn + qx - q^3 t + \eta^0\right)$$

where p and η^0 are constants, and the parameters λ, μ given by $\lambda = q/(1+q)$, $\mu = -q$ and $\gamma = 0$. Further by applying the BT (18) and (19) to the 1-soliton solution $f(n) = 1 + \exp(\eta_1)$ we can deduce the following 2-soliton solution

$$g(n) = 1 + A_1 e^{\eta_1} + e^{\eta_2} + A_2 e^{\eta_1+\eta_2}$$

where

$$\eta_i = p_i n + q_i x - q_i^3 t + \eta_i^0$$

$$A_1 = \frac{q_2 + q_1}{q_2 - q_1} \qquad A_2 = -\frac{\sinh(\frac{1}{2}(p_1 - p_2))}{\sinh(\frac{1}{2}(p_1 + p_2))}$$

with p_i, q_i and η_i^0 constants for the set of parameters $\lambda = q_2/(1+q_2)$, $\mu = -q_2$ and $\gamma = 0$. Besides, by using MATHEMATICA, we can show that (9) has the 3-soliton solution

$$f = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_3} + A_{12}\mathrm{e}^{\eta_1+\eta_2} + A_{13}\mathrm{e}^{\eta_1+\eta_3} + A_{23}\mathrm{e}^{\eta_2+\eta_3} + A_{12}A_{23}A_{13}\mathrm{e}^{\eta_1+\eta_2+\eta_3}$$

where

$$\eta_i = p_i n + q_i x - q_i^3 t + \eta_i^0 \qquad A_{ij} = \frac{(q_i - q_j)\sinh(\frac{1}{2}(p_i - p_j))}{(q_i + q_j)\sinh(\frac{1}{2}(p_i + p_j))}.$$

Similar to those in section 2, based on the permutability of the BT (18) and (19), we can also derive the following nonlinear superposition formula for (9):

$$\left(\mathrm{e}^{-\frac{1}{2}D_n} - \frac{1}{\lambda_1\lambda_2}\mathrm{e}^{\frac{1}{2}D_n}\right) f_0 \bullet f_{12} = c\left(\lambda_1 \mathrm{e}^{\frac{1}{2}D_n} - \lambda_2 \mathrm{e}^{-\frac{1}{2}D_n}\right) f_1 \bullet f_2$$

with c being a nonzero constant.

In the following, we are going to derive a Lax pair for (11) from the BT (18) and (19). For this purpose, we set $\psi(n) = g(n)/f(n)$, $u(n) = (\ln f(n))_x$. Then from the bilinear BT (18) and (19), we have

$$(\lambda\psi(n+1) - \psi(n))_x = (u(n) - u(n+1))(\psi(n) + \lambda\psi(n+1)) + \mu(\psi(n) - \lambda\psi(n+1)) \quad (20)$$

$$\psi_t(n) = -3\mu\psi_{xx}(n) - 6\mu u_x(n)\psi(n) - \psi_{xxx}(n) - 6u_x(n)\psi_x(n) - 3\mu^2\psi_x(n) + \gamma\psi(n). \quad (21)$$

In analogy to section 2, we introduce $T_1 = \lambda T_+ - 1$, $T_2 = 1 + \lambda T_+$ and $\psi_1 = T_1\psi(n)$, $\psi_2 = T_2\psi(n)$, where T_+ is a shift operator defined by $T_+u(n) \equiv u(n+1)$. Then (20) and (21) become

$$\psi_{1x} = -(u(n+1) - u(n))\psi_2 - \mu\psi_1 \quad (22)$$

$$\begin{aligned}\psi_{1t} = {} & -3\mu\psi_{1xx} - \psi_{1xxx} - 3\mu(u_x(n+1) - u_x(n))\psi_2 - 3\mu(u_x(n+1) + u_x(n))\psi_1 \\ & -3(u_x(n+1) - u_x(n))\psi_{2x} - 3(u_x(n+1) + u_x(n))\psi_{1x} + \gamma\psi_1 - 3\mu^2\psi_{1x}.\end{aligned} \quad (23)$$

By some calculations and using relations

$$\begin{aligned}&T_1\psi_2 = T_2\psi_1\\ &T_1(a(n)b(n)) = \tfrac{1}{2}(b(n+1) - b(n))T_2a(n) + \tfrac{1}{2}(b(n+1) + b(n))T_1a(n)\\ &T_2(a(n)b(n)) = \tfrac{1}{2}(b(n+1) + b(n))T_2a(n) + \tfrac{1}{2}(b(n+1) - b(n))T_1a(n)\end{aligned}$$

we can derive (11) from the compatibility condition: $(\psi_1)_{xt} = (\psi_1)_{tx}$. To summarize, we obtain the following result:

Proposition 4. *Equations (20) and (21) constitute a Lax pair for (11).*

Besides, we can obtain the following Lie symmetries for (9):

$$\begin{aligned}&\sigma_1 = h(t)f\\ &\sigma_2 = xh(t)f\\ &\sigma_3 = h(t)f_x - \tfrac{1}{12}\dot{h}(t)x^2 f\\ &\sigma_4 = h(t)f_t + \tfrac{1}{3}x\dot{h}(t)f_x - \tfrac{1}{108}\ddot{h}(t)x^3 f\end{aligned}$$

where $h(t)$ is an arbitrary function of t and $\dot{h}(t) \equiv \frac{\mathrm{d}h(t)}{\mathrm{d}t}$, $\ddot{h}(t) \equiv \frac{\mathrm{d}^2h(t)}{\mathrm{d}t^2}$.

4. Conclusions and discussion

Two bilinear differential–difference equations have been proposed. Their corresponding bilinear BTs have been presented. Starting from the bilinear BTs, soliton solutions are generated. By dependent variable transformations, these two equations are transformed into equations in nonlinear variables and the corresponding Lax pairs are derived. Besides, we can also consider the continuous analogues of these two differential–difference equations. For example, we look at the differential–difference equation (11) and expand $u(n+1)$ as

$$u(n+1) = u + \epsilon \frac{\partial}{\partial y} u + \frac{\epsilon^2}{2} \frac{\partial^2}{\partial y^2} u + \frac{\epsilon^3}{6} \frac{\partial^3}{\partial y^3} u + \cdots \tag{24}$$

$$u_t(n+1) = u_t + \epsilon \frac{\partial^2}{\partial y \partial t} u + \frac{\epsilon^2}{2} \frac{\partial^3}{\partial y^2 \partial t} u + \frac{\epsilon^3}{6} \frac{\partial^4}{\partial y^3 \partial t} u + \cdots \tag{25}$$

$$u_x(n+1) = u_x + \epsilon \frac{\partial^2}{\partial y \partial x} u + \frac{\epsilon^2}{2} \frac{\partial^3}{\partial y^2 \partial x} u + \frac{\epsilon^3}{6} \frac{\partial^4}{\partial y^3 \partial x} u + \cdots . \tag{26}$$

Substituting (24)–(26) etc into (11) and neglecting higher-order terms of ϵ, we obtain

$$u_{yt} + u_{xxxy} + 6u_x u_{yx} + 6u_{xx} u_y = 0 \tag{27}$$

which is nothing but the Ito equation [10] with $y \leftrightarrow t$. In fact, by the dependent variable transformation $u = (\ln f)_x$, we can transform (27) into bilinear form

$$D_y(D_t + D_x^3) f \bullet f = 0. \tag{28}$$

Obviously, when $y = x$, (27) becomes the KdV equation by integration.

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Appendix

The following bilinear operator identities hold for arbitrary functions a, b and c:

$$\begin{aligned}&[D_t \sinh(\tfrac{1}{2}D_n) a \bullet a][\exp(\tfrac{1}{2}D_n) b \bullet b] - [D_t \sinh(\tfrac{1}{2}D_n) b \bullet b][\exp(\tfrac{1}{2}D_n) a \bullet a] \\ &\qquad = 2\sinh(\tfrac{1}{2}D_n)(D_t a \bullet b) \bullet ab\end{aligned} \tag{A.1}$$

$$\begin{aligned}&[D_x^3 \sinh(\tfrac{1}{2}D_n) a \bullet a][\exp(\tfrac{1}{2}D_n) b \bullet b] - [D_x^3 \sinh(\tfrac{1}{2}D_n) b \bullet b][\exp(\tfrac{1}{2}D_n) a \bullet a] \\ &\qquad = 2\sinh(\tfrac{1}{2}D_n)(D_x^3 a \bullet b) \bullet ab - 3D_x[D_x \exp(\tfrac{1}{2}D_n) a \bullet b] \bullet [D_x \exp(-\tfrac{1}{2}D_n) a \bullet b]\end{aligned} \tag{A.2}$$

$$\begin{aligned}&D_x[(D_x \exp(\tfrac{1}{2}D_n) a \bullet b) \bullet (\exp(-\tfrac{1}{2}D_n) a \bullet b) + (\exp(\tfrac{1}{2}D_n) a \bullet b) \bullet (D_x \exp(-\tfrac{1}{2}D_n) a \bullet b)] \\ &\qquad = 2\sinh(\tfrac{1}{2}D_n)(D_x^2 a \bullet b) \bullet ab\end{aligned} \tag{A.3}$$

$$\sinh(\delta D_n) c \bullet c = 0 \tag{A.4}$$

$$D_x c \bullet c = 0. \tag{A.5}$$

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